Applied Mathematics Letters 22 (2009) 1364–1368

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Fixed point theorems for two new classes of multivalued mappings

Mujahid Abbas [a], B.E. Rhoades [b,*]

[a] Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, Pakistan

[b] Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, United States

ARTICLE INFO

Article history:
Received 6 August 2008
Received in revised form 3 April 2009
Accepted 6 April 2009

Keywords:
Weak*-nonexpansive multivalued mappings
Generalized *RM* mappings
Banach space
Metric spaces

ABSTRACT

The notions of weak*-nonexpansive mapping and generalized R-multivalued mapping are introduced. Fixed points, common fixed points and some properties of the set of fixed points (common fixed points) of these maps are studied.

1. Introduction and preliminaries

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [1]. Later, an interesting and rich fixed point theory for such maps was developed. The theory of multivalued maps has application in control theory, convex optimization, differential equations and economics. The concept of *-nonexpansive multivalued maps has been introduced and studied by Husain and Latif [2] which is a generalization of the usual notion of nonexpansiveness for single-valued maps. It has been demonstrated by Khan and Hussain [3] that a *-nonexpansive multivalued map is not weakly nonexpansive and in general, *-nonexpansive multivalued maps are neither nonexpansive nor continuous. Recently, Suzuki [4] introduced a class of single-valued mappings satisfying a condition which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In this paper we have extended this notion to multivalued mappings, and have introduced a new class of multivalued mappings called weak*-nonexpansive maps, which includes the class of weakly nonexpansive multivalued maps, and have obtained a fixed point result. The notion of a generalized R-multivalued mapping is also introduced and a fixed point theorem for this class of mappings is proved. These results extend and unify various comparable results from the literature [2,5–8].

Consistently with [9], we use the following notation for in a metric space (X, d); $x \in X, A \subseteq X$, and $d(x, A) = \inf\{d(x, A) : y \in A\}$. We denote by 2^X, and $CB(X)$ the families of all nonempty, and nonempty bounded and closed subsets of X, respectively. Let H be the Hausdorff metric induced by d of X, and given by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for every $A, B \in CB(X)$. Let C be a subset of X. We recall that a multivalued mapping $T : C \longrightarrow 2^X$ is said to be nonexpansive if

$$H(Tx, Ty) \le d(x, y).$$

The mapping T is said to be demiclosed at x^* in X if $\{x_n\}$ is a sequence in C which converges weakly to x and $y_n \in Tx_n \to x^*$; then $x^* \in Tx$.

* Corresponding author.
E-mail addresses: mujahid@lum.edu.pk (M. Abbas), rhoades@indiana.edu (B.E. Rhoades).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.04.002

Definition 1.1. A mapping $T : C \longrightarrow 2^X$ is called weakly nonexpansive if, for each $x \in C$ and $u_x \in Tx$, there is a u_y in Ty for each $y \in C$ such that

$$d(u_x, u_y) \leq d(x, y).$$

An example of a weakly nonexpansive map is given by

$$Tx = \cup_{\alpha \in I} T_\alpha x,$$

where I is an indexing set and T_α is the family of single-valued nonexpansive mappings of C into itself [10].

Definition 1.2. A mapping $T : C \longrightarrow 2^X$ is called *-nonexpansive if, for each $x \in C$ and $u_x \in Tx$ with

$$d(x, u_x) = d(x, Tx),$$

there is a u_y in Ty with

$$d(y, u_y) = d(y, Ty)$$

such that

$$d(u_x, u_y) \leq d(x, y).$$

The notions of *-nonexpansive multivalued maps and weakly nonexpansive multivalued maps are independent of each other [10].

We introduce a new class of multivalued mappings.

Definition 1.3. Let C be a subset of a Banach space X. A mapping $T : C \longrightarrow 2^X$ is called weak*-nonexpansive if, for each $x \in C$, and $u_x \in Tx$ and $y \in C$ for which

$$\frac{1}{2} d(x, u_x) \leq d(x, y), \tag{1}$$

there exists a u_y in Ty such that

$$d(u_x, u_y) \leq d(x, y). \tag{2}$$

We give an example of a nonexpansive mapping which is not weakly nonexpansive.

Example 1.4. Let $T : [0, 1] \to 2^{[0,1]}$ be a multivalued mapping defined by

$$Tx = \begin{cases} \{1\} & \text{when } x \in \left[0, \frac{1}{2}\right) \\ \left[0, \frac{x}{2}\right] & \text{when } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Note that, if we take $x = \frac{1}{2}$, $u_x = 0 \in Tx$, then for $y = 0$, we have $u_y = 1$ and $d(u_x, u_y) = 1 > d(x, y)$.

Obviously a weak*-nonexpansive mapping need not be *-nonexpansive. The following is an example of a *-nonexpansive mapping which is not weak*-nonexpansive, which establishes the fact that the definitions of weak*-nonexpansive and *-nonexpansive are independent.

Example 1.5. Define $T : [0, \infty) \to 2^{[0,\infty)}$ by

$$Tx = [x, 2x].$$

Note that T is a *-nonexpansive mapping [10]. However for each x in X, choose $y \in X$ so that $0 < y < \frac{x}{2}$. Then

$$\frac{1}{2} d(x, u_x) = \frac{1}{2}(u_x - x) \leq \frac{x}{2} < x - y = d(x, y).$$

But for $u_x = 2x$,

$$d(u_x, u_y) = 2x - u_y \geq 2(x - y) > x - y = d(x, y),$$

and T is not weak*-nonexpansive.

Note that a weakly nonexpansive mapping is weak*-nonexpansive. The following is an example of a weak*-nonexpansive which is not weakly nonexpansive.

Example 1.6. Let $X = [0, \infty)$, $C = [0, 3] \cup [7, 8] \subseteq X$, and $T : C \to 2^X$ be a multivalued mapping defined by

$$Tx = \begin{cases} (0, x] - \{1\} & \text{when } x \in [0, 3] \\ \{0\} & \text{when } x \in [7, 8) \\ \{1\} & \text{when } x = 8. \end{cases}$$

If $x \in [7, 8)$ and $y = 8$ or $y \in [7, 8)$ and $x = 8$, then (1) is not satisfied. Direct calculations verify that, for all x, y satisfying (1), condition (2) is also satisfied. To show that T is not weakly nonexpansive, let $x = 8 - \varepsilon$ for ε small, and $y = 8$. Then $d(u_x, u_y) = 1 > d(x, 8)$ and (2) is not satisfied.

Now we establish a fixed point theorem for a weak*-nonexpansive multivalued mapping.

Theorem 1.7. *Let C be a weakly compact subset of a Banach space X and $T : C \longrightarrow CB(C)$ a weak*-nonexpansive multivalued mapping. If $I - T$ is demiclosed at 0, then T has a fixed point.*

Proof. Let $x_1 \in C$ and $x_{n+1} \in \lambda Tx_n + (1 - \lambda)x_n$ for $n \in N$, and λ a constant satisfying $\lambda \in [\frac{1}{2}, 1)$. This implies that

$$x_{n+1} = \lambda y_n + (1 - \lambda)x_n$$

for some $y_n \in Tx_n$. Now

$$\frac{1}{2}d(x_n, Tx_n) \le \frac{1}{2}\|x_n - y_n\| \le \lambda \|x_n - y_n\| = \|x_n - x_{n+1}\|$$

for $n \in N$. Since T is weak*-nonexpansive, there exists a y_{n+1} in Tx_{n+1} such that

$$\|y_n - y_{n+1}\| \le \|x_n - x_{n+1}\| .$$

Using Lemma 3 [4] (see also, [11]), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Since C is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to x^* in C. Also, $z_{n_k} = x_{n_k} - y_{n_k} \in I - Tx_{n_k}$ and $z_{n_k} \to 0$. It follows from the demiclosedness of $I - T$ at 0 that $0 \in I - Tx^*$; that is, $x^* \in Tx^*$, as desired. □

Kannan [6] has proved a fixed point theorem for a single-valued self-mapping T of X satisfying the property

$$d(Tx, Ty) \le h\{d(x, Tx) + d(y, Ty)\}$$

for all x, y in X and for a fixed $h \in [0, \frac{1}{2})$. Latif and Beg [7] introduced the notion of a K-multivalued mapping, which is the extension of Kannan mappings to multivalued mappings. Recently, Rus et al. [8] coined the term R-multivalued mapping, which is a generalization of a K-multivalued mapping.

Definition 1.8. Two multivalued maps $S, T : X \longrightarrow 2^X$ are called a generalized R-multivalued pair if (i) for each $x, y \in X$, $u_x \in Sx$, there exists a $u_y \in Ty$ such that (3) holds, and (ii) for each $x, y \in X$, $u_y \in Ty$, there exists a $u_x \in Sx$ such that (3) holds, and

$$d(u_x, u_y) \le h \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d(x, u_y) + d(y, u_x)}{2}\right\}, \tag{3}$$

where $0 \le h < 1$.

Using a two-map analogue of (3) we prove the following result.

Theorem 1.9. *Let (X, d) be a complete metric space, $S, T : X \longrightarrow CB(X)$ a generalized R-multivalued pair satisfying (3). Then $F(S) = F(T) \neq \phi$. Moreover, $F(S) = F(T)$ is closed.*

Proof. Let $x^* \in X$ be a fixed point of S; that is $x^* \in Sx^*$. Then by hypothesis, there exists an $x \in Tx^*$ such that

$$\begin{aligned} d(x^*, x) &\le h \max\left\{d(x^*, x^*), d(x^*, x^*), d(x, x^*), \frac{d(x^*, x) + d(x^*, x^*)}{2}\right\} \\ &= h \max\left\{d(x, x^*), \frac{d(x, x^*)}{2}\right\} = hd(x, x^*), \end{aligned}$$

which gives $x^* = x$; that is, $F(S) \subseteq F(T)$. Similarly, $F(T) \subseteq F(S)$. Hence $F(S) = F(T)$. Let $x_0 \in X$ and $x_1 \in Sx_0$. Then by hypothesis there exists an $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\le h \max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\} \\ &= h \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\} \\ &= h \max\{d(x_0, x_1), d(x_1, x_2)\} = hd(x_0, x_1), \end{aligned}$$

since $h < 1$. Similarly, there exists an $x_3 \in Sx_2$ such that

$$d(x_2, x_3) \leq hd(x_1, x_2).$$

By induction, there exists a sequence $\{x_m\}$ with $x_0 \in X$, $x_{2m-1} \in Sx_{2m-2}$, and $x_{2m} \in Tx_{2m-1}$ such that

$$d(x_m, x_{m+1}) \leq h^m d(x_0, x_1). \tag{4}$$

In obtaining (4) it has been assumed that $x_n \neq x_{n+1}$ for each n. Such an assumption is reasonable. For suppose that there is an integer N such that $x_N = x_{N+1}$. If N is odd, then there is some m for which $N = 2m - 1$ and we have $x_{2m-1} = x_{2m} \in Sx_{2m-1}$, and $x_{2m-1} \in F(S) = F(T)$. Similarly, if N is even, then $x_{2m} \in F(T) = F(S)$. From (4) it follows that $x_m \to x^* \in X$ as $m \to \infty$, for some $x^* \in X$. Further, $x_{2m} \in Tx_{2m-1}$ and so there exists a u_m in Sx^* such that

$$d(u_m, x_{2m}) \leq h \max\left\{d(x^*, x_{2m-1}), d(x^*, u_m), d(x_{2m}, x_{2m-1}), \frac{d(x^*, x_{2m}) + d(x_{2m-1}, u_m)}{2}\right\}. \tag{5}$$

Since $\{u_m\}$ is bounded, $\limsup_{m\to\infty} u_m = u^*$, and $\liminf_{m\to\infty} u_m = u_*$ both exist. Taking the lim sup of both sides of (5) yields $u^* = x^*$. Similarly, taking the lim inf of both sides gives $u_* = x^*$. Since $u_m \in Sx^*$ for all $n \geq 1$ and Sx^* is a closed set, it follows that $x^* \in Sx^*$.

Since $F(S) = F(T) \neq \phi$, let $x_n \in F(S)$, $n \in N$, be such that $x_n \to x^*$ as $n \to \infty$. Since $x_n \in Sx_n$ for each n, by definition, there exists a $v_n \in Tx^*$ such that

$$d(x_n, v_n) \leq h \max\left\{d(x_n, x^*), d(x_n, x_n), d(x^*, v_n), \frac{d(x_n, v_n) + d(x^*, x_n)}{2}\right\}.$$

Note that $\{v_n\}$ is bounded. A similar argument implies that $d(x^*, v_n) \to 0$ as $n \to \infty$. Since $v_n \in Tx^*$ for all $n \geq 1$ and Tx^* is a closed set, it follows that $x^* \in Tx^*$ and hence $x^* \in Sx^*$, which establishes the result. □

Example 1.10. Let $X = [0, 1]$ with the usual metric. Define $S, T : X \to CB(X)$ by

$$Sx = \left[0, \frac{x}{4}\right], \qquad Tx = \left[0, \frac{x}{2}\right]$$

for all $x \in X$.

We shall first show that, for each $x \in X$, each $u_x \in Sx$, and for each $y \in X$, there exists some $u_y \in Ty$ such that (3) is satisfied.
Case I. $0 \leq x < y$. Choose $u_y = 0$. Then

$$d(u_x, u_y) \leq u_x \leq \frac{x}{4}.$$

$$\begin{aligned} m(x, y) &= \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d(x, u_y) + d(y, u_x)}{2}\right\} \\ &= \max\left\{y - x, x - u_x, y, \frac{x + y - u_x}{2}\right\} \geq x - u_x \geq \frac{3x}{4} \end{aligned}$$

and (3) is satisfied with $h = \frac{1}{3}$.
Case II. $0 \leq y \leq x$. The same conclusion holds with $u_y = 0$.

$$d(u_x, u_y) \leq u_x \leq \frac{x}{4}.$$

$$\begin{aligned} m(x, y) &= \max\left\{y - x, x - u_x, y, \frac{x + y - u_x}{2}\right\} \\ &\geq x - u_x \geq \frac{3x}{4} \end{aligned}$$

and (3) is satisfied with $h = \frac{1}{3}$.
We now wish to show that, for each $y \in X$, each $u_y \in Ty$, and for each $x \in X$, there exists some $u_x \in Sx$ such that (4) is satisfied.
Case III. Assume that $0 \leq x < y$. (a) For $u_y \leq \frac{1}{4}$, choose $u_x = u_y$. Then (3) is trivially satisfied. (b) For $\frac{1}{4} < u_y$, choose $u_x = \frac{u_y}{2}$. Then

$$d(u_x, u_y) = \frac{u_y}{2} \leq \frac{y}{4}.$$

$$\begin{aligned} m(x, y) &= \max\left\{y - x, |x - u_x|, y - u_y, \frac{|x - u_x| + y - u_y}{2}\right\} \\ &\geq y - u_y \geq \frac{y}{2} \end{aligned}$$

and (3) is satisfied with $h = \frac{1}{2}$.

Case IV. Assume that $0 \le y \le x$. Choose $u_x = 0$. Then

$$d(u_x, u_y) = u_y \le \frac{y}{2}.$$

$$\begin{aligned} m(x, y) &= \max\left\{x - y, x, y - u_y, \frac{x - u_y + y}{2}\right\} \\ &\ge \frac{x - u_y + y}{2} \ge \frac{2y - \frac{y}{2}}{2} = \frac{3y}{4} \end{aligned}$$

and (3) is satisfied with $h = \frac{2}{3}$.
Therefore (3) is satisfied with $h = \frac{2}{3}$.

Acknowledgements

The authors are grateful to the referees for their precise remarks which led to improvement of the presentation of the paper.

References

[1] J.T. Markin, Continuous dependence of fixed point sets, Proc. Amer. Math. Soc. 38 (1973) 545–547.
[2] T. Husain, A. Latif, Fixed points of multivalued nonexpansive maps, Math. Japonica 33 (1988) 385–391.
[3] A.R. Khan, N. Hussain, Random fixed points for *-nonexpansive random operators, J. Appl. Math. Stoc. Anal. 14 (4) (2001) 341–349.
[4] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2) (2008) 1088–1095.
[5] T. Husain, E. Tarafdar, Fixed point theorems for multivalued mappings of nonexpansive type, Yokohama Math. J. 28 (1980) 1–6.
[6] R. Kannan, Some results on fixed points, Bull. Calcutta. Math. Soc. 60 (1968) 71–76.
[7] A. Latif, I. Beg, Geometric fixed points for single valued and multivalued mappings, Demonstratio Math. 30 (4) (1997) 791–800.
[8] I.A. Rus, A. Petrusel, A. Sintamarian, Data dependence of the fixed point set of some multivalued weakly Picard operators, Nonlinear Anal. 52 (2003) 1947–1959.
[9] S.B. Nadler Jr., Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488.
[10] H.K. Xu, On weakly nonexpansive and *-nonexpansive multivalued mappings, Math. Japonica 36 (3) (1991) 441–445.
[11] K. Goebel, W.A. Kirk, Iteration processes for nonexpansive mappings, Contemp. Math. 21 (1983) 115–123.